Grading guide, Pricing Financial Assets, August 2016

1. Assume that the price of a non-dividend paying stock S can be modelled (under the original probability measure $\mathbb{P}$) by the geometric Brownian motion

$$dS = \mu S dt + \sigma S dz$$

where μ and $\sigma > 0$ are constants, and where dt and dz are the standard short hand notations for a small time-step and a Brownian increment.

(a) Describe the qualitative characteristics of this model, and discuss its possible shortcomings as a model of a stock.

(b) Consider the transformation G of S given by the natural logarithm (ln), i.e. $G(x) = \ln(x)$. Use Ito's lemma to find the process followed by $G(S)$.

(c) Consider a forward contract on the stock maturing at time T. Compare the pricing of a call option on the stock with a pricing of a call on the forward contract on the stock both maturing at T and having the same strike K.

Solution:

(a) The answer should cover an interpretation of the parameters of the model, and note the resulting continuous sample paths and the lognormal distribution of prices or the normal distribution of returns. The student should note that realized sample paths - especially for a single stock - may show large jumps (e.g. at the publication of new information), and that realized returns empirically tend to show heteroscedasticity at odds with the normal distribution (Hull, esp. sections 13.3-4 and 14.1-5). The student may also note that empirical return data tend to show negative skewness and excess kurtosis, but this is not directly covered by the syllabus (but at the lectures and in Hull e.g. 19.3 and 26.1).

(b) The use of Ito's lemma is straightforward (on this transformation, see Hull section 13.7).

(c) First the answer should note the definiton of a forward contract, in particular that it has no payments before maturity.

Then the answer should cover the forward price that will be agreed barring arbitrage: With an assumption of a constant risk free interest rate this is for the forward price F_t at contracted at time $t \leq T$, where T is the maturity of the contract

$$F_t = \mathsf{E}_{\mathbb{Q}}\left[S_T\right] = S_t \mathsf{e}^{r(T-t)}$$

, where we only for the last equation use the assumption of no dividends.

Last the answer should note that F follows a Geometric Brownian Motion similar to that derived earlier for S, but that the drift rate of returns is $\mu - r$ rather than the μ for the stock. Under the $\mathbb{Q}$-measure the drift rates will be 0 and r, respectively. Thus comparing two call options with the same strike, but where the first is on the forward contract and the second is on the stock,

$$c_t^F = \mathsf{e}^{-r(T-t)}\mathsf{E}_{\mathbb{Q}}\left[max[F_T - K; 0]\right] = \mathsf{e}^{-r(T-t)}\mathsf{E}_{\mathbb{Q}}\left[max[S_T - K; 0]\right] = c_t^S$$

since they will have the same payout at maturity, even if the forward price initially (with a positive interest rate) is higher than the spot price (It is not required that this observation is formulated using formulas, as long as the student makes the argument based on the identity of payouts at maturity). This may be verified by looking at the formulae for call prices on stocks and forwards (if you e.g. substitute the relation between forward and spot prices into the Black formula you get the Black-Scholes formula for a call), (see Hull 17.3).

2. Consider European call and put options on the same futures contract. Both options have strike price K and expiry at time T. Assume that there is a constant risk free interest rate of r. Let the futures price at time t be F_t (You may ignore margining and treat the futures as forward contracts).

   (a) At time 0 derive the put-call parity for the futures options.

   (b) Use the put-call parity to find the relationship between

      1. The delta of the European futures call and the delta of the European futures put (use the delta with respect to the futures price rather than the spot price)
      2. The gamma of the European futures call and the gamma of the European futures put (use the gamma with respect to the futures price rather than the spot price)
      3. The vega of the European futures call and the vega of the European futures put
      4. The theta of the European futures call and the theta of the European futures put

      and comment on the results.

Solution:

(a) Let $t \leq T$ and let c_t and p_t be the prices of the call and the put, respectively.

Consider the portfolio (A) of the call and $\mathsf{e}^{-r(T-t)}K$ placed at interest and the portfolio (B) of a put, a long position in the future with expiry at T and $\mathsf{e}^{-r(T-t)}F_t$ placed at interest.

At maturity the portfolios will have the payouts:

$$A : max[F_T - K; 0] + K$$
$$B : max[K - F_T; 0] + (F_T - F_t) + F_t$$

which are identical.

Barring arbitrage we thus must have

$$c_t + \mathsf{e}^{-r(T-t)}K = p_t + \mathsf{e}^{-r(T-t)}F_t$$

See Hull section 17.4.

(b) The student is required to define the relevant Greeks. Then the results follow directly from the above by partial differentiation, e.g. for Theta you get:

$$\Theta_c - \Theta_p = r\mathsf{e}^{-r(T-t)}(F_t - K)$$

i.e. the Thetas are identical for an at-the-money call and similar put and generally close for low interest rates.

3. The HJM-model describes the simultaneous evolution of the full term structure of interest rates. Let the evolution of instantaneous forward rates contracted at t for time T be described by the Ito-process

$$df(t,T) = m(t,T,\Omega)dt + s(t,T,\Omega)dz$$

where Ω is a set of state variables, and dt and dz are the standard shorthand notations for a small time step and a Brownian increment.

   (a) Under certain conditions we have the following no-arbitrage condition for the drift term:

$$m(t,T,\Omega) = s(t,T,\Omega)\int_t^T s(t,\tau,\Omega)d\tau$$

Comment on this result, and in particular explain under which probability measure it is derived.

(b) As a special case let $s(t,T,\Omega)$ be a constant denoted s. Derive the process followed by forward rates. Comment on the distribution of the forward rates.

(c) Certain models of the term structure can be calibrated to be consistent with an initial given term structure. How is this achieved in simple one-factor models as e.g. the Ho-Lee model? How is this achieved in the HJM-model?

Solution:

(a) Cf. Hull, pp. 716-7. What should be commented is that the volatility barring arbitrage determines the drift rate, and that the result presented is derived under the traditional risk neutral probability measure.

(b) By integration we find for this simple model that

$$df(t,T) = s^2(T-t)dt + sdz$$

This means that the changes to the forward rates, and thus the forward rates, will be normally distributed, which in particular means a positive probability for negative interest rates. It can be noted - but this is not directly in the syllabus (it is in Practice Question 31.3) - that this special case is equivalent to the Ho-Lee-model.

(c) The models that can be made consistent with any (arbitrage free, i.e. having no negative forward rates) initial term structure are termed no-arbitrage models (Hull p. 689ff). This is in models of the short rate achieved by making the drift term dependent on time, e.g. the Ho-Lee model and other named models (Hull p. 690f). In the HJM-model this is directly achieved as the initial value of the full term structure of forward rates (In either case these instantanous rates are not directly observable in the market and must be derived from prices on traded instruments).